

Revisiting the deconfinement phase transition in SU(4) Yang-Mills theory in 2+1 dimensions

Kieran Holland

*Department of Physics, University of the Pacific,
3601 Pacific Ave, Stockton CA 95211, U.S.A.
E-mail: kholland@pacific.edu*

Michele Pepe

*INFN, Istituto Nazionale di Fisica Nucleare, Sezione di Milano-Bicocca,
Edificio U2, Piazza della Scienza, 3 - 20126 Milano, Italy
E-mail: pepe@mib.infn.it*

Uwe-Jens Wiese

*Institute for Theoretical Physics, Bern University,
Sidlerstrasse 5, CH-3012 Bern, Switzerland
E-mail: wiese@itp.unibe.ch*

ABSTRACT: In order to deepen our understanding of the nature of the deconfinement phase transition for various gauge groups, we investigate SU(4) Yang-Mills theory in 2 + 1 dimensions. We find that the transition is weakly first order. We perform extensive Monte Carlo simulations on lattices with temporal extent $N_t = 3, 4$, and 5, and spatial sizes up to $N_s = 20 N_t$. We observe coexistence of confined and deconfined phases at the critical temperature, and finite-size scaling shows consistency with first order exponents. The continuum extrapolation of the latent heat yields $L_h/T_c^3 = 0.188(17)$.

KEYWORDS: Spontaneous Symmetry Breaking, Lattice Gauge Field Theories.

Contents

| | |
|---|-----------|
| 1. Motivation | 1 |
| 2. SU(4) Yang-Mills theory on the lattice | 3 |
| 2.1 The action and the observables | 3 |
| 2.2 Finite-size scaling | 5 |
| 3. Discussion of the numerical results | 6 |
| 3.1 Simulation details | 6 |
| 3.2 Monte Carlo histories and Polyakov loop distributions | 6 |
| 3.3 Polyakov loop susceptibility | 8 |
| 3.4 Specific and latent heats | 9 |
| 4. Conclusions | 11 |

1. Motivation

The confinement of quarks inside baryons and mesons is a feature of the strong interactions only at low temperatures. At sufficiently high temperatures hadrons melt and quarks and gluons form a plasma. Numerical simulations of QCD support the expectation that this process takes place in a smooth manner and the high- and low-temperature regimes are analytically connected through a crossover. As one increases or decreases the quark masses, the situation changes and, at some point, a finite temperature phase transition occurs. This phase transition is related to symmetries that are either badly broken or only approximate when the quarks have their physical masses. For massless quarks, chiral symmetry is exact: it is spontaneously broken at low temperatures and it gets restored in a chiral phase transition at finite temperature. In the opposite limit, when the quarks are heavy, they are only weakly coupled to the gluons. As the quark mass becomes much larger than the typical energy scale of the strong interactions, $\Lambda_{\text{QCD}} \approx 250 \text{ MeV}$, the quarks decouple and gluons are the only relevant degrees of freedom. In the limit of infinitely heavy quarks, the global center symmetry of Yang-Mills theory is no longer explicitly broken by the quarks' triality and it becomes an exact symmetry of the theory. When the temperature is about 300 MeV, the center symmetry breaks spontaneously at a phase transition [1–3]. Since quarks transform non-trivially under center transformations, the breaking of the center symmetry implies deconfinement.

Yang-Mills theory provides a simplified framework in which the phenomenon of confinement can be investigated without facing the more difficult numerical problems related

to dynamical fermions. Moreover, the pure gluon dynamics is quite rich and many investigations have been performed to study various interesting features of Yang-Mills theory. For instance, string effects in the static quark potential have been observed numerically [4]. The study of topological objects and of their relevance for the mechanism of confinement is an active field of investigation [5]. A systematic study of $SU(N)$ Yang-Mills theory for various N is a research topic that aims at understanding the way in which the large N limit is approached [6, 7]. This paper deals with another important characteristic of Yang-Mills theory, namely the order of the deconfinement phase transition.

About 25 years ago, Svetitsky and Yaffe conjectured [8] that the critical behavior of a gauge theory at the deconfinement transition can be described by a scalar field theory with a symmetry corresponding to the center of the gauge group. In fact, if one integrates out the spatial components of the gluon field in $(d + 1)$ -dimensional Yang-Mills theory, one obtains an effective action for the scalar field represented by the Polyakov loop. The corresponding scalar field theory is defined in d dimensions and, in general, its action is very complicated. However, if the deconfinement phase transition happens to be second order, as one approaches the critical point, the correlation length diverges and universal critical behavior arises. Hence, the details of the complicated effective action become irrelevant: only the center symmetry and the dimensionality of space determine the universality class.

Svetitsky and Yaffe's conjecture has been checked in many numerical simulations in Yang-Mills theory with various gauge groups, both in $2+1$ and in $3+1$ dimensions. In those cases in which the deconfinement phase transition is second order, the universality class has indeed turned out to be the one predicted by Svetitsky and Yaffe. In $3+1$ dimensions, $SU(2)$ Yang-Mills theory has a second order deconfinement phase transition [9–14] in the universality class of the 3-dimensional Ising model [15, 16]. However, in $3+1$ dimensions no other pure gauge theory has been found to have a second order deconfinement phase transition [17–26]. In $2+1$ dimensions, $SU(2)$ Yang-Mills theory again has a second order deconfinement phase transition, now in the universality class of the 2-dimensional Ising model [27, 28]. Since in $2+1$ dimensions fluctuations are stronger than in $3+1$ dimensions, there are two more cases in which the deconfinement phase transition is second order. At its deconfinement phase transition, $(2+1)$ -dimensional $SU(3)$ Yang-Mills theory shows the same critical behavior as the 2-dimensional 3-state Potts model [28–30]. The group $Sp(2)$ has the same center $\mathbb{Z}(2)$ as $SU(2)$, and the deconfinement phase transition of the corresponding $(2+1)$ -d $Sp(2)$ Yang-Mills theory is again in the universality class of the 2-dimensional Ising model [31]. For other gauge groups, the deconfinement phase transition of the corresponding pure gauge theory has turned out to be of first order [32, 33].

In this paper, we examine $SU(4)$ Yang-Mills theory in $2+1$ dimensions because it is one of the last remaining unsettled cases. The original study on coarse lattices indicated that the transition is second order [34]. The improved numerical results presented in [35] show that, on coarse lattices, the transition is weakly first order. For finer lattices, the deconfinement phase transition appeared to be second order, perhaps belonging to the universality class of the 2-dimensional 4-state Potts model. This is a particular case of the 2-d $\mathbb{Z}(4)$ -symmetric Ashkin-Teller model which has lines of critical points along which the universality class and the critical exponents change continuously. The authors of [35]

pointed out that it is difficult to obtain a definite answer unless one considers rather large volumes and they could not rule out a weak first order phase transition in the continuum limit. The need for large volumes and fine lattices was also emphasized in [33].

In this paper we present numerical evidence for a weak first order deconfinement phase transition in SU(4) Yang-Mills theory in $2 + 1$ dimensions. Interestingly, although there is an infinite set of different available universality classes, the system deconfines with non-universal behavior. The center symmetry does not play a role in determining the order of the deconfinement phase transition. Only when the transition is second order does the center symmetry determine the universality class. As we conjectured in [31], the order of the deconfinement phase transition is determined by the size of the group. In the low-temperature confined phase, the dynamics of Yang-Mills theory is governed by glueballs. The number of glueball states — i.e. the number of singlets in the tensor product decomposition of adjoint representations — is essentially independent of the gauge group. On the other hand, the dynamics of the high-temperature plasma phase is determined by deconfined gluons, whose number is given by the number of generators of the gauge group. If there is a large mismatch between the number of relevant degrees of freedom in the confined and the deconfined phases, the phase transition does not proceed smoothly as a second order transition. Instead an abrupt discontinuous first order transition takes place. This conjecture is supported by numerical simulations which show that the strength of the first order transition increases with the size of the gauge group. Further evidence was provided by studies of Yang-Mills theory with the exceptional gauge group $G(2)$ [36]. The group $G(2)$ is the smallest, simply connected group with a trivial center. Therefore, in $G(2)$ Yang-Mills theory there is no symmetry argument that implies the presence of a finite temperature deconfinement phase transition. However, since $G(2)$ (which has 14 generators) has a rather large size, $G(2)$ Yang-Mills theory has a first order deconfinement phase transition in $3 + 1$ dimensions [37–39]. Various aspects concerning the problem of confinement in $G(2)$ Yang-Mills theory have been investigated in [40, 41].

The rest of the paper is organized as follows. In section 2, we describe the standard lattice formulation of Yang-Mills theory, the observables that we consider, and the finite-size scaling analysis used to determine the order of the phase transition. The numerical results are presented in section 3, followed by our conclusions.

2. SU(4) Yang-Mills theory on the lattice

2.1 The action and the observables

We perform numerical simulations of SU(4) Yang-Mills theory on a periodic lattice in $2 + 1$ dimensions. We consider the standard Wilson plaquette action

$$S[U] = -\frac{\beta}{4} \sum_{\square} \text{ReTr } U_{\square} = -\frac{\beta}{4} \sum_{x,\mu < \nu} \text{ReTr } (U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu}^{\dagger} U_{x,\nu}^{\dagger}), \quad (2.1)$$

where the link parallel transporter matrices $U_{x,\mu} \in \text{SU}(4)$ are group elements in the fundamental representation. All dimensionful quantities are expressed in units of the lattice

spacing. The bare dimensionful gauge coupling β is related to the usual gauge coupling g in the continuum by $\beta = 8/g^2$. The path integral measure and the partition function Z then take the form

$$\int \mathcal{D}U = \prod_{x,\mu} \int_{\text{SU}(4)} dU_{x,\mu}, \quad Z = \int \mathcal{D}U \exp(-S[U]). \quad (2.2)$$

The Polyakov loop [1, 2]

$$\Phi_{\vec{x}} = \frac{1}{4} \text{Tr} \left(\mathcal{P} \prod_{t=1}^{N_t} U_{\vec{x},t,d+1} \right) \quad (2.3)$$

is the trace of a path-ordered product of link variables along a loop wrapping around the periodic Euclidean time direction. Here $N_t = 1/T$ is the extent of the lattice in the Euclidean time direction, which determines the temperature T in lattice units. The expectation value of the Polyakov loop is given by

$$\langle \Phi \rangle = \frac{1}{Z} \int \mathcal{D}U \frac{1}{N_s^2} \sum_{\vec{x}} \Phi_{\vec{x}} \exp(-S[U]), \quad (2.4)$$

where N_s is the extension of the lattice in the spatial directions. The Polyakov loop represents a scalar field that transforms non-trivially under symmetry transformations in the center subgroup $\mathbb{Z}(4)$ of $\text{SU}(4)$. Hence, a non-vanishing expectation value of the Polyakov loop indicates the spontaneous breakdown of the center symmetry and thus signals deconfinement. However, in a finite periodic volume spontaneous symmetry breaking — in the sense of a non-vanishing order parameter — cannot occur. Therefore, in the finite-size scaling analysis discussed below, we will consider the expectation value of the magnitude of the Polyakov loop $\langle |\Phi| \rangle$. In a finite volume this quantity is always non-vanishing but it approaches zero when one takes the thermodynamic limit in the confined phase. Another quantity that is useful for distinguishing the confined from the deconfined phase is the probability distribution for the Polyakov loop,

$$p(\Phi) = \frac{1}{Z} \int \mathcal{D}U \delta \left(\Phi - \frac{1}{N_s^2} \sum_{\vec{x}} \frac{1}{4} \text{Tr} \left(\mathcal{P} \prod_{t=1}^{N_t} U_{\vec{x},t,d+1} \right) \right) \exp(-S[U]). \quad (2.5)$$

In the confined phase $p(\Phi)$ has a single peak centered at $\Phi = 0$. In the deconfined phase, on the other hand, it has four degenerate maxima at $\Phi = \Phi_0 \exp(ik\pi/2)$, where Φ_0 is a positive real number and $k = 0, 1, 2, 3$. When the deconfinement phase transition is first order, the confined and the deconfined phases coexist and can be distinguished by their different values of the Polyakov loop even at the phase transition. In that case, close to the phase transition one thus observes five maxima of the distribution $p(\Phi)$. The relative weight of the confined and deconfined peaks changes as one crosses the phase transition. On the other hand, when the deconfinement phase transition is second order, the high- and low-temperature phases become indistinguishable at criticality. The confined maximum becomes broader and broader as the critical temperature is approached from below and, at criticality, the width of the peak diverges. When the temperature is increased further, the four $\mathbb{Z}(4)$ -symmetric deconfined peaks emerge smoothly from the broad distribution of

the Polyakov loop. In the limit of very high temperatures, one can perform an analytic perturbative calculation of the effective potential for the Polyakov loop [42–46].

Other useful observables that characterize the deconfinement phase transition are the Polyakov loop susceptibility χ , defined by

$$\chi = N_s^2 (\langle |\Phi|^2 \rangle - \langle |\Phi| \rangle^2), \quad (2.6)$$

and the specific heat C given by

$$C = \frac{1}{3N_s^2 N_t} (\langle S^2 \rangle - \langle S \rangle^2). \quad (2.7)$$

For a first order transition it is interesting to also consider the latent heat L_h . As we said above, in the case of a first order transition one can distinguish the confined from the deconfined phases even at the transition. One can then define the action densities s_c and s_d for the confined and the deconfined phases, respectively. The latent heat is defined as the difference in the action density between the two phases at the critical temperature and it is given by

$$L_h = s_d - s_c. \quad (2.8)$$

The fluctuations in the action attain their maximum when the two phases have the same probability. It then follows that, in the thermodynamic limit, the maximum C^{\max} of the specific heat is given by

$$C^{\max} = \frac{3N_s^2 N_t}{4} L_h^2. \quad (2.9)$$

For a second order deconfinement phase transition the latent heat vanishes since the confined and deconfined phases become indistinguishable at the critical point.

2.2 Finite-size scaling

Away from a phase transition, the susceptibility of an extensive quantity scales with the volume. This scaling behavior changes as we approach a phase transition and the fluctuations become stronger. For a first order transition, the susceptibility of an extensive quantity increases with the square of the volume. This scaling behavior follows from the coexistence of the two phases. In fact, in general, an observable has different values in the two phases. Since the observable is extensive, its susceptibility scales with the square of the volume. In case of a second order phase transition, the susceptibility scales faster than the volume but — unlike for a first order transition — not as fast as the square of the volume. The exponent that characterizes the scaling behavior depends on the observable and on the universality class of the phase transition.

The method we have used in the finite-size scaling analysis of our numerical data is the following. For a given lattice size $N_s^2 \times N_t$, we perform a set of numerical simulations at various couplings β across the deconfinement phase transition. Using the Ferrenberg-Swendsen re-weighting technique [47, 48] we determine a pseudo-critical coupling β_{c,N_s,N_t} from the maximum of the Polyakov loop susceptibility. We then repeat this procedure for

various values of the spatial lattice size N_s . For a first order phase transition the critical coupling depends on the spatial volume as

$$\beta_{c,N_s,N_t} = \beta_{c,N_t} + a_0 \frac{N_t^2}{N_s^2} + \dots \quad (2.10)$$

where β_{c,N_t} is the critical coupling in the limit of infinite spatial volume at fixed temporal extent N_t .

Up to corrections to scaling, the data for the Polyakov loop susceptibility density χ/N_s^2 collected at different couplings β and for different lattice sizes $N_s^2 \times N_t$ collapse onto a single universal curve once they are plotted as a function of the finite-size scaling variable $x = (N_s/N_t)^2 (\beta/\beta_{c,N_s,N_t} - 1)$. The corrections to scaling can then be easily measured at the maximum of the curve

$$\frac{\chi^{\max}}{N_s^2} = \left(\frac{\chi^{\max}}{N_s^2} \right)_{\infty} + b_0 \frac{N_t^2}{N_s^2} + \dots \quad (2.11)$$

A similar formula holds for the maximum of the specific heat

$$\frac{C^{\max}}{3N_s^2 N_t} = \frac{1}{4} L_h^2 + c_0 \frac{N_t^2}{N_s^2} + \dots \quad (2.12)$$

3. Discussion of the numerical results

3.1 Simulation details

We have performed simulations on lattices with $N_t = 3, 4$, and 5 , and for spatial sizes N_s as large as $20 N_t$. We have used a standard combination of heat-bath [49] and over-relaxation [50–53] algorithms to update the various SU(2) subgroups of SU(4) [54]. We have simulated with a ratio of over-relaxation to heat-bath updates of $4/1$ and $1/1$, and we find no significant difference between these two choices. For each set of β, N_t , and N_s , we have generated at least 10^5 configurations to be used for measurements. These runs are sufficiently long such that for the smaller physical volumes with e.g. $N_s/N_t = 10$, we see $\mathcal{O}(50-100)$ tunneling events between the various bulk phases. For the larger volumes like $N_s/N_t = 20$, this is reduced to $\mathcal{O}(10)$ tunnelings. We expect that this is a reasonable sampling of the various bulk phases.

3.2 Monte Carlo histories and Polyakov loop distributions

In figure 1, we plot the Monte Carlo histories of the Polyakov loop and the plaquette expectation value, configuration by configuration, for a $60^2 \times 3$ lattice at $\beta = 20.40$. The system spends a long time in a particular phase, characterized by the value of Φ , before it rapidly tunnels to a different phase, in which it again remains for a significant period of Monte Carlo time. In this particular run, we see the confined phase, in which Φ fluctuates around 0, and four deconfined phases, in which Φ varies around the four values $\Phi_0 \exp(ik\pi/2)$. The lower plot shows that the plaquette changes simultaneously with the Polyakov loop, between two similar but still distinguishable values. This suggests that the system is close

| N_t | N_s | β_{c,N_s,N_t} |
|-------|-------|---------------------|
| 3 | 26 | 20.251(8) |
| | 28 | 20.271(6) |
| | 30 | 20.288(7) |
| | 36 | 20.3230(25) |
| | 42 | 20.351(5) |
| | 48 | 20.363(4) |
| | 60 | 20.390(4) |
| | 72 | 20.388(5) |
| 4 | 32 | 25.966(14) |
| | 36 | 26.019(14) |
| | 40 | 26.065(9) |
| | 48 | 26.097(10) |
| | 52 | 26.135(12) |
| | 56 | 26.173(10) |
| | 64 | 26.203(11) |
| | 80 | 26.192(9) |
| 5 | 34 | 31.64(4) |
| | 40 | 31.75(4) |
| | 46 | 31.72(4) |
| | 50 | 31.77(3) |
| | 56 | 31.91(4) |
| | 60 | 31.977(26) |
| | 66 | 32.012(24) |
| | 100 | 32.090(22) |

Table 1: The finite-volume critical couplings for various values of N_s and N_t , with bootstrap errors.

to the phase transition, and that the deconfinement transition is first order, with coexisting confined and deconfined phases. However, the small jump in the plaquette value indicates that the transition may well be rather weak. This simulation is quite typical in that we see about 10 tunneling events occur in this large volume.

In figure 2, we show the probability distributions of the complex-valued Polyakov loop for simulations on $36^2 \times 3$ lattices at three different β values, ranging from low to high temperature. At low temperature, there is just the confined phase, while at high temperature there are four deconfined phases. At $\beta = 20.26$, the system is apparently quite close to the transition temperature and the five bulk phases coexist. Figure 3 shows the probability distributions of the plaquette value for the same simulations. We find only a single-peak distribution, which varies smoothly with β . In the case of a normal-strength first order transition, close to the critical temperature one would expect to see two distinct peaks. In the present case, the discontinuity in the plaquette value is clearly visible in the Monte Carlo history — as shown in the bottom part of figure 1 — but due to its small

size, it does not stand out in the plot of the probability distribution.

3.3 Polyakov loop susceptibility

We use re-weighting of the various ensembles to determine the location β_{c,N_s,N_t} of the peak of the Polyakov loop susceptibility χ . This is an accurate method and we show typical results in figures 4 and 5. We use the bootstrap method to calculate both the error in χ for each individual ensemble, and the error in β_{c,N_s,N_t} extracted from the re-weighting of several ensembles. We list the finite-volume critical couplings in table 1 for the various simulations we have performed.

If the deconfinement transition is of first order, the infinite-volume critical coupling should be approached asymptotically as $(N_t/N_s)^2$, as described in Equation (2.10). The data as shown in figures 6, 7, and 8 display exactly this behavior. In determining the infinite-volume critical coupling for the various N_t values, there is some systematic error involved in the choice of the extrapolation range. One does not know how large N_s has to be before linear behavior sets in. Let us first discuss the $N_t = 3$ data. We start by fitting all of the data, then omit one at a time the data for the smaller values of N_s . The results are listed in table 2, with a statistical error included for each fit. For $N_t = 3$, we see that the fit does not improve as data is omitted, so the optimal fit includes all of them. For the final error estimate, we use the jackknife method applied to the optimal data set. We quote a final result of $\beta_{c,N_t} = 20.414(5)$. The optimal fit and this extrapolated value are also shown in figure 6.

We apply exactly the same procedure to the $N_t = 4$ results. From figure 7, we see that a linear fit of all of the data gives $\beta_{c,N_t} = 26.251(13)$, the error being statistical only. However, the quality of the fit is poor, with $\chi^2/\text{d.o.f.} = 17.4/6$. Omitting the data for smaller N_s , the quality of the fit does not improve, in fact it becomes worse, as listed in table 2. The small N_s data do not seem to be at fault, as significant fluctuations occur at larger N_s . A quadratic fit gives $\beta_{c,N_t} = 26.26(3)$ and $\chi^2/\text{d.o.f.} = 17.2/5$, so the data show almost no quadratic behavior and the extrapolated value agrees very well with the linear fit. Hence, the optimal choice seems to be to use all of the data. Performing a jackknife error analysis on this set, the final result we quote is $\beta_{c,N_t} = 26.251(16)$.

For both $N_t = 3$ and 4, a linear extrapolation appears valid for a fitting range $(N_t/N_s)^2 \leq 0.015$. We expect that the same should hold for $N_t = 5$. Looking at the data in figure 8, two of the data lie outside this range. At sufficiently small N_s , one does expect to see deviations from linearity. A linear fit of all of the data gives a very poor quality of fit. When the two smallest N_s values are omitted, $\chi^2/\text{d.o.f.}$ improves significantly, from 26.1/6 to 7.8/4, giving $\beta_{c,N_t} = 32.22(4)$, the error being statistical only. As listed in table 2, the fit improves further if the next two small N_s data are discarded. However, we believe that this is excessive and gives the impression of a more accurate extrapolation than is warranted. Using the linear regimes of $N_t = 3$ and 4 as a guide, we conclude that the optimal fit excludes only the two smallest N_s data. For comparison, we also fit all of the data using a quadratic form, which gives $\beta_{c,N_t} = 32.24(6)$ and $\chi^2/\text{d.o.f.} = 14.9/5$. The extrapolated value is completely consistent with that from a linear fit, although the

| N_t | N_{data} | β_{c,N_t} | $\chi^2/\text{d.o.f.}$ |
|-------------|-------------------|-----------------|------------------------|
| 3 | 8 | 20.414(4) | 6.9/6 |
| | 7 | 20.415(4) | 6.2/5 |
| | 6 | 20.416(5) | 5.5/4 |
| | 5 | 20.418(6) | 4.8/3 |
| | 4 | 20.416(10) | 4.6/2 |
| | 3 | 20.417(18) | 4.5/1 |
| final value | | 20.414(5) | |
| 4 | 8 | 26.251(13) | 17.4/6 |
| | 7 | 26.253(16) | 17.0/5 |
| | 6 | 26.253(20) | 17.0/4 |
| | 5 | 26.26(3) | 16.2/3 |
| | 4 | 26.24(3) | 8.0/2 |
| | 3 | 26.22(3) | 2.8/1 |
| final value | | 26.251(16) | |
| 5 | 8 | 32.14(4) | 26.1/6 |
| | 7 | 32.18(4) | 16.2/5 |
| | 6 | 32.22(4) | 7.8/4 |
| | 5 | 32.21(5) | 7.4/3 |
| | 4 | 32.164(21) | 0.69/2 |
| | 3 | 32.153(5) | 0.02/1 |
| final value | | 32.22(8) | |

Table 2: The infinite-volume critical couplings for the various values of N_t and the quality of the linear extrapolations, where some of the smaller N_s data are discarded.

quadratic fit is somewhat inferior. Using the jackknife method on the optimal set, we quote a final value for $N_t = 5$ of $\beta_{c,N_t} = 32.22(8)$.

We also investigate the finite-size scaling behavior of the Polyakov loop susceptibility. In figures 9, 10, and 11, we plot the rescaled susceptibility χ/N_s^2 as a function of the finite-size scaling variable, $x = (N_s/N_t)^2 (\beta/\beta_{c,N_s,N_t} - 1)$, for $N_t = 3, 4$, and 5. We are assuming here that the exponents are those of a first order transition. We see that, close to the critical temperature, the data do indeed collapse onto a single universal curve. This is further evidence of the first order nature of the deconfinement transition.

3.4 Specific and latent heats

Besides the Polyakov loop, we wish to use another thermodynamic quantity in order to determine the order of the phase transition. We have attempted to measure the latent heat L_h by simulating directly at the appropriate pseudo-critical coupling β_{c,N_s,N_t} . Based on the value of the Polyakov loop, we have divided each ensemble into confined and deconfined configurations. We have then measured the action in each phase and have determined the discontinuity. Unfortunately, this method has some difficulties due to the somewhat arbitrary cut in the value of the Polyakov loop used to distinguish confined from deconfined configurations. We have found that it is more accurate to measure C^{max} , the peak in the specific heat.

| N_t | N_s | $C^{\max}/(3N_s^2N_t) \times 10^4$ |
|-------|-------|------------------------------------|
| 3 | 24 | 13.66(7) |
| | 26 | 11.81(6) |
| | 28 | 10.22(6) |
| | 30 | 8.95(6) |
| | 36 | 6.46(4) |
| | 42 | 4.89(3) |
| | 48 | 3.84(4) |
| | 54 | 3.13(3) |
| | 60 | 2.69(4) |
| 4 | 32 | 3.722(17) |
| | 36 | 2.960(16) |
| | 40 | 2.407(15) |
| | 48 | 1.69(10) |
| | 52 | 1.446(8) |
| | 56 | 1.255(12) |
| | 64 | 0.977(12) |
| | 80 | 0.635(13) |
| 5 | 34 | 1.990(10) |
| | 40 | 1.436(6) |
| | 46 | 1.091(5) |
| | 50 | 0.925(4) |
| | 56 | 0.739(3) |
| | 60 | 0.646(3) |
| | 66 | 0.5352(29) |
| | 100 | 0.2406(26) |

Table 3: The finite-volume maxima of the specific heat for various values of N_s and N_t , with bootstrap errors.

In table 3, we list the finite-volume maxima of the specific heat for $N_t = 3, 4$, and 5. The quoted errors for each ensemble are calculated using the bootstrap method. In figures 12, 13, and 14 we plot the data for $C^{\max}/(3N_s^2N_t)$ for $N_t = 3, 4$, and 5 respectively, and for various values of N_s . It is no surprise that the data do indeed extrapolate accurately in $(N_t/N_s)^2$, as a first order transition dictates. The infinite-volume values $C^{\max}/(3N_s^2N_t)(\infty)$ are quite small, and accurate data are needed in order to reach a reliable conclusion. Like the finite-volume critical couplings, one does not know *a priori* how large N_s has to be before the linear regime in $(N_t/N_s)^2$ is reached. We find that omitting the smaller N_s data does not improve the already very good linear fits, so all of the data are used in the final analysis. The results of the fits are presented in table 4, where the errors are calculated by the jackknife method.

| N_t | $C^{\max}/(3N_s^2 N_t)(\infty)$ | $\chi^2/\text{d.o.f.}$ |
|-------|---------------------------------|------------------------|
| 3 | $5.66(22) \times 10^{-5}$ | 5.8/8 |
| 4 | $5.5(5) \times 10^{-6}$ | 0.65/6 |
| 5 | $1.19(11) \times 10^{-6}$ | 0.42/6 |

Table 4: The extrapolated infinite-volume maxima of the specific heat, and the quality of the linear fits. Errors are calculated using the jackknife method.

Using Equation (2.12), we convert the infinite-volume extrapolations into the latent heat L_h . We extrapolate the dimensionless quantity $N_t^3 L_h = L_h/T_c^3$ to the continuum linearly in $1/N_t^2$, which describes the data very well. Our continuum determination is $L_h/T_c^3 = 0.188(17)$, where the error is statistical, and the quality of the fit is $\chi^2/\text{d.o.f.} = 0.90/1$. With only three data points, it is not possible to estimate a systematic error by varying the choice of fitting range. The transition clearly becomes weaker on finer lattices, but its first order nature persists in the continuum limit.

4. Conclusions

Our results show that $(2 + 1)$ -d $SU(4)$ Yang-Mills theory has a first order deconfinement phase transition. Large and fine lattices were important for reaching this result. In the various extrapolations and universal curves, our assumption of using first order exponents is well confirmed by the data. The first order nature of the transition is further supported by the observed coexistence of the confined and deconfined phases for all three temporal extensions $N_t = 3, 4$, and 5 that we have considered. It is much harder to independently determine the critical exponents than to show consistency with an expected set. Since the 2-d $\mathbb{Z}(4)$ -symmetric Ashkin-Teller model has continuously varying critical exponents, the challenge is particularly large. In [35], by including logarithmic corrections to scaling, the data suggested a second order deconfinement transition belonging to the universality class of the 2-d 4-state Potts model. The numerical data presented here suggest that this is not the case. The determination of a non-zero latent heat in the limit of vanishing lattice spacing shows that the deconfinement phase transition does not weaken to second order but stays first order in the continuum limit.

It is surprising that so few gauge theories realize the Svetitsky-Yaffe scenario which only applies when the deconfinement phase transition is second order. This may be particularly surprising in the present case, in which the infinite set of different universality classes of the 2-d Ashkin-Teller model would be available. In $3 + 1$ dimensions, $SU(2) = Sp(1)$ Yang-Mills theory is the only pure gauge theory with a second order deconfinement phase transition. In $2 + 1$ dimensions, the transition is second order only for $SU(2)$, $SU(3)$, and $Sp(2)$ Yang-Mills theory. Even though all symplectic groups $Sp(N)$ have the same center $\mathbb{Z}(2)$, the transition becomes first order as N increases, in both $3 + 1$ and $2 + 1$ dimensions. Interestingly, the 3-d $\mathbb{Z}(N)$ -symmetric spin model belongs to the universality class of the $U(1)$ -symmetric XY model for $N \geq 5$, i.e. the symmetry is enhanced at the critical point [55]. However, the corresponding $(3 + 1)$ -d $SU(N)$ gauge theory is unaffected

by this peculiar critical behavior because its deconfinement transition is first order of a strength increasing with N . Indeed, as we conjectured in [31], not the center but the size of the gauge group determines the order of the deconfinement transition. In $3 + 1$ dimensions all Yang-Mills theories whose gauge group has more than three generators have first order transitions. As we now know, in $2 + 1$ dimensions only the Yang-Mills theories whose gauge group has at most ten generators (namely $SU(2)$, $SU(3)$, and $Sp(2)$, which has ten generators), have a second order deconfinement phase transition. An interesting case for future study is $(2+1)$ -d $G(2)$ Yang-Mills theory. Since the exceptional group $G(2)$ has a trivial center, there is no symmetry reason for a deconfinement phase transition and there may hence just be a crossover. In the absence of a non-trivial center, a second order phase transition can be ruled out on theoretical grounds, because it would require unnatural fine-tuning of some parameter. If the 14 deconfined $G(2)$ gluons at high temperature cannot smoothly crossover to the low-temperature regime governed by a small number of glueball states, $(2 + 1)$ -d $G(2)$ Yang-Mills should have a first order deconfinement phase transition. Since the 15 $SU(4)$ gluons behave in this way, based on the size of the gauge group, one may expect the same for $G(2)$.

Acknowledgments

We wish to thank Philippe de Forcrand for useful discussions and Urs Wenger for providing us with his code for the re-weighting technique. K.H. is supported by the National Science Foundation under grant NSF 0704171, and would also like to thank the University of the Pacific for their support via the Eberhardt Research Fellowship. This work is also supported in part by the Schweizerischer Nationalfonds. The simulations were performed on a Dell cluster at the University of the Pacific, and on a PC cluster at the University of Milano-Bicocca.

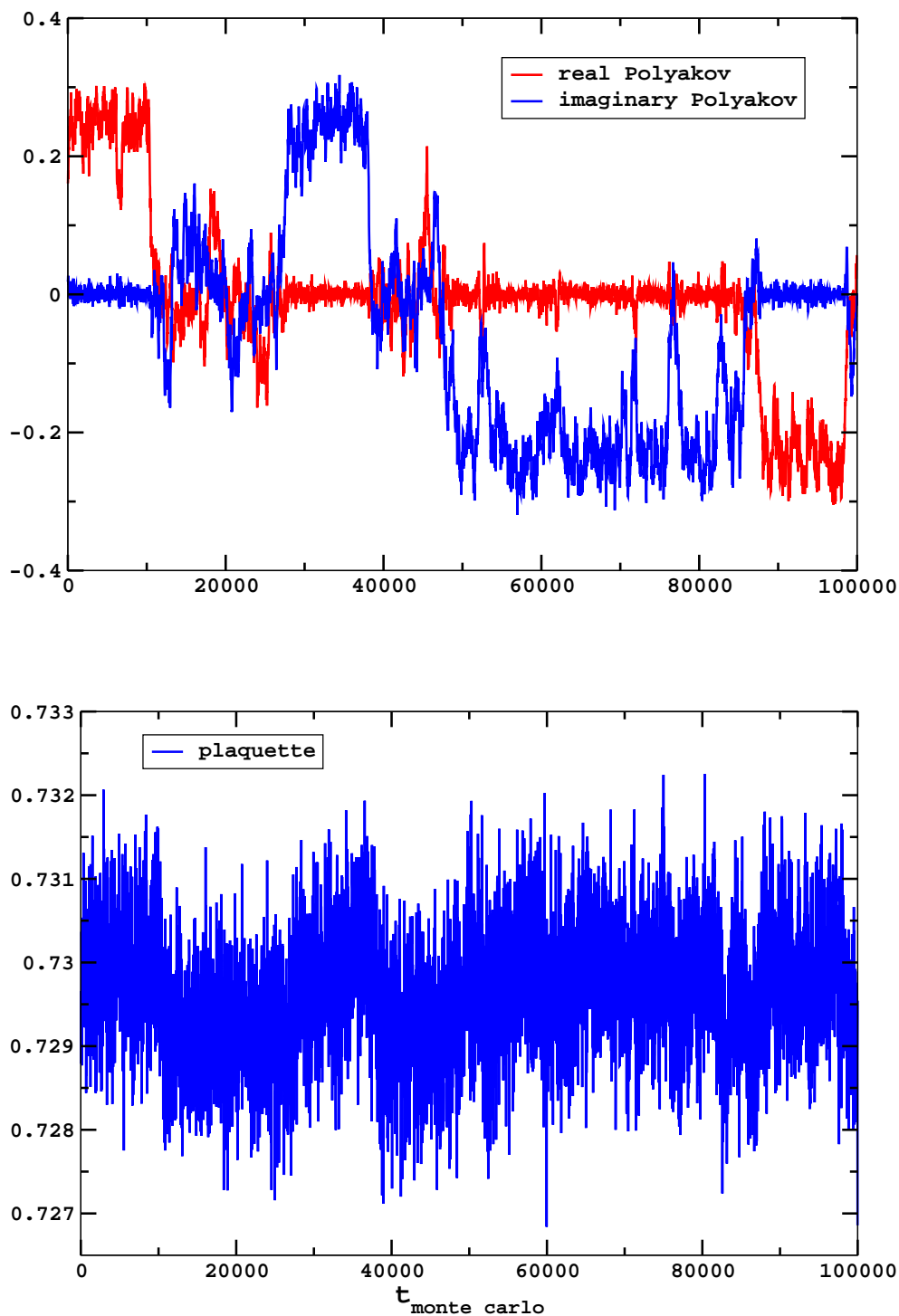


Figure 1: The Monte Carlo histories of the Polyakov loop and the plaquette on a $60^2 \times 3$ lattice at $\beta = 20.40$, close to the deconfinement transition. The system tunnels between the confined and four deconfined phases, with the plaquette value tracking the change in the Polyakov loop.

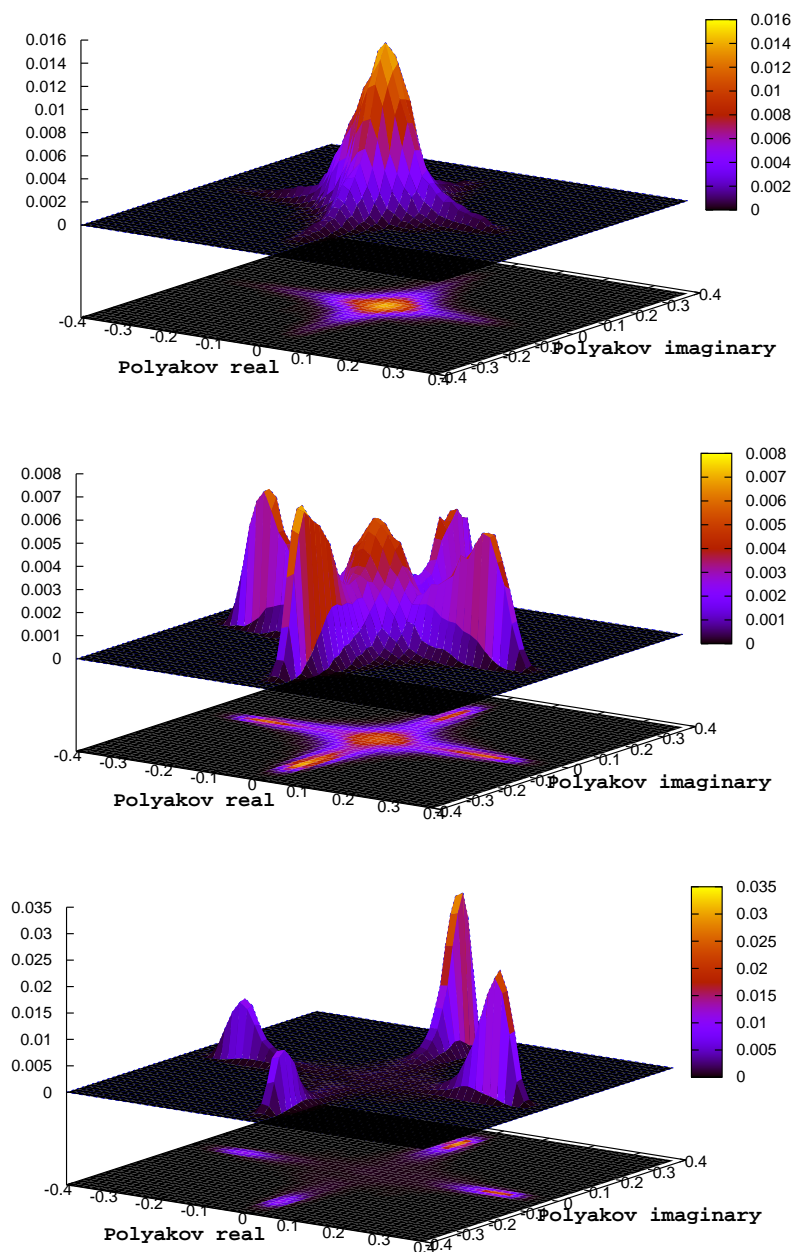


Figure 2: The probability distributions of the Polyakov loop on $36^2 \times 3$ lattices at $\beta = 20.0$ (top), 20.26 (middle) and 20.5 (bottom). At low temperature, there is a single confined phase. Close to the critical temperature, we observe coexistence of the confined with the four deconfined phases. At high temperature, there are only the four deconfined phases. Because of the finite length of the simulation, the four deconfined phases are not equally sampled.

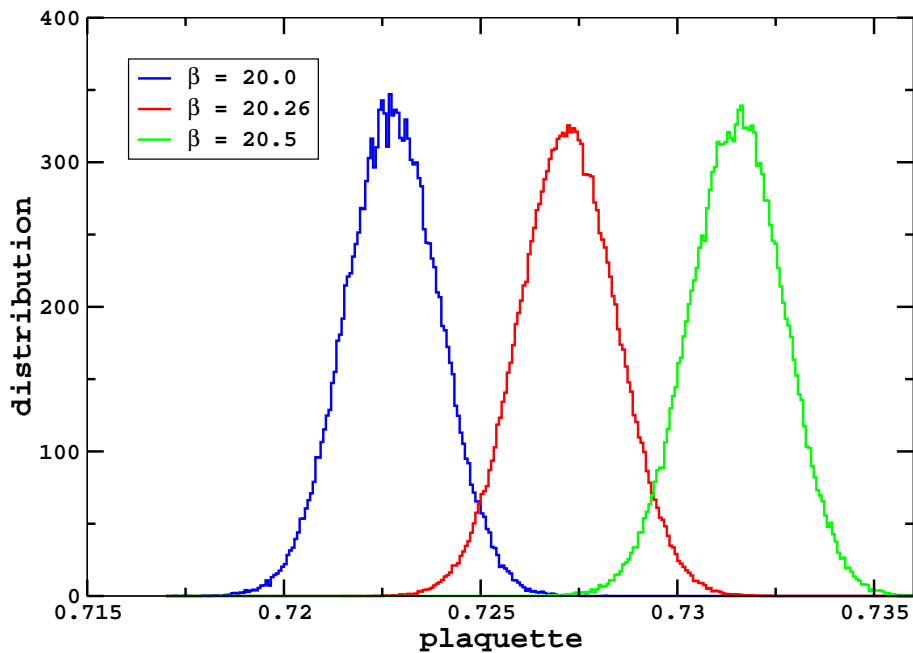


Figure 3: The plaquette value distributions for the $36^2 \times 3$ simulations shown in figure 2. Close to the critical temperature, at $\beta = 20.26$, we do not find two well-separated peaks, as one would expect for a normal-strength first order phase transition.

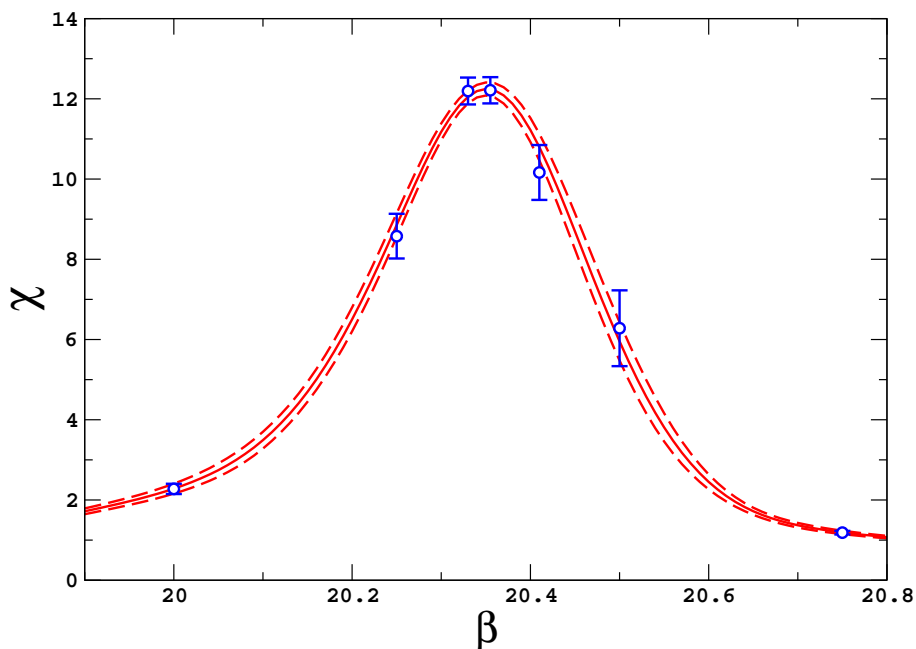


Figure 4: The Polyakov loop susceptibility χ as a function of β for $42^2 \times 3$ lattices, using re-weighting of the combined ensembles. The critical coupling, where χ has a maximum, is determined to be $\beta_{c,N_s,N_t} = 20.351(5)$.

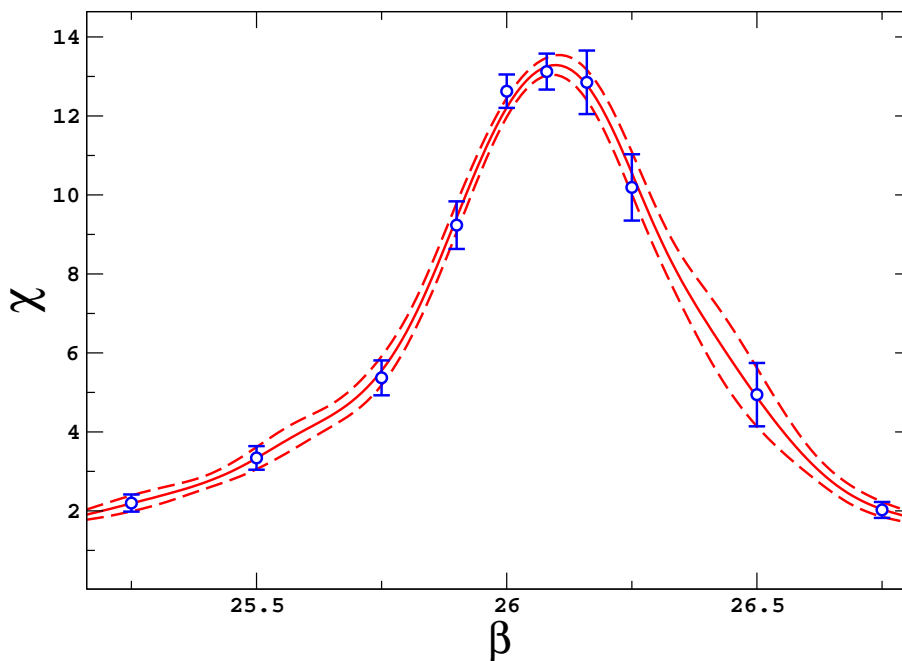


Figure 5: The same as figure 4, for $48^2 \times 4$ lattices. The critical coupling is $\beta_{c,N_s,N_t} = 26.097(10)$.

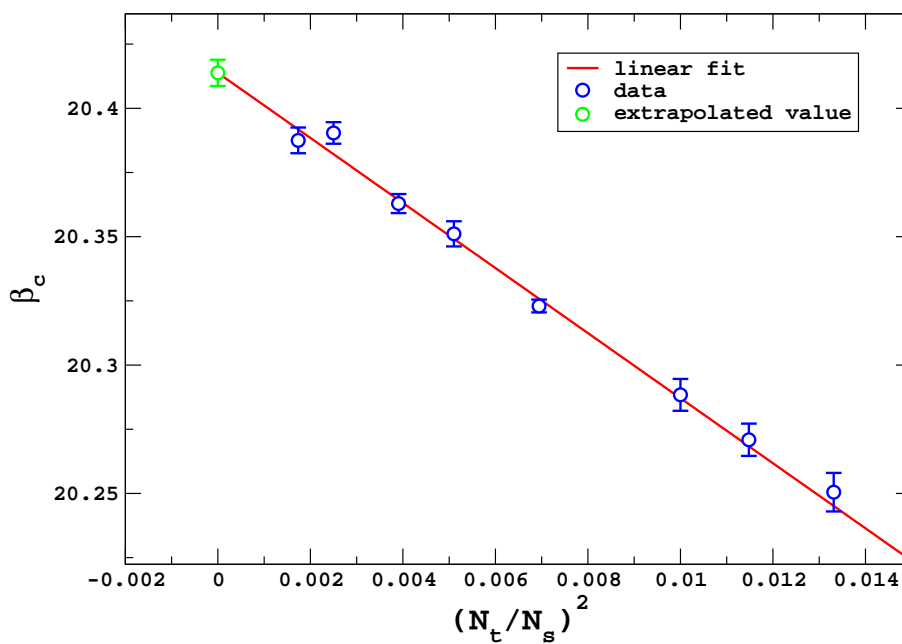


Figure 6: The critical couplings β_{c,N_s,N_t} for $N_t = 3$ and a range of N_s , extracted from the peak of the Polyakov loop susceptibility χ . A linear extrapolation in $(N_t/N_s)^2$ describes the data well, as expected for a first order transition. The infinite-volume extrapolated value is $\beta_{c,N_t} = 20.414(5)$ (jackknife error) and the quality of the fit is $\chi^2/\text{d.o.f.} = 6.9/6$.

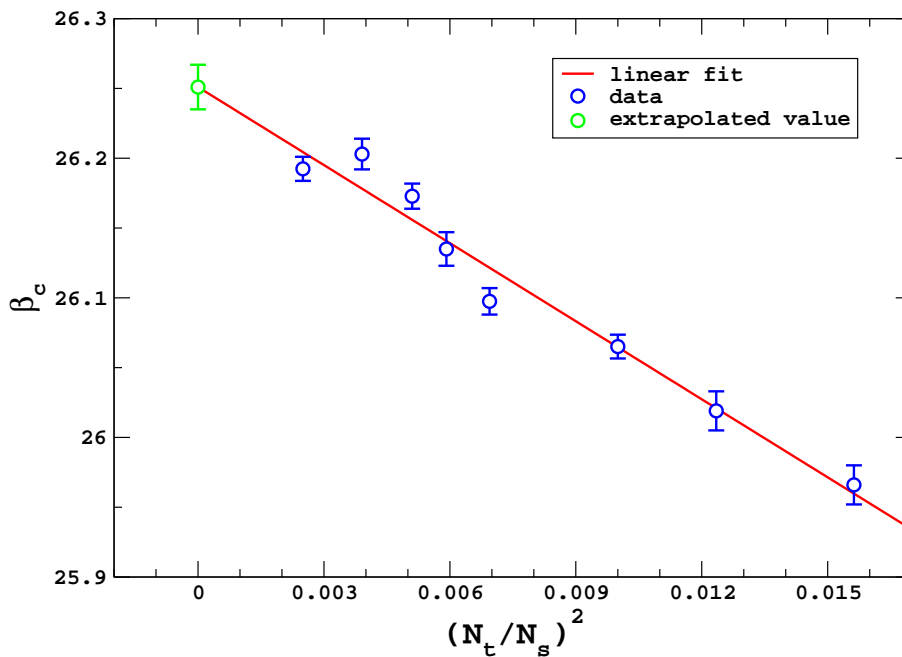


Figure 7: The same as in figure 6, this time for $N_t = 4$. The extrapolated value is $\beta_{c,N_t} = 26.251(16)$ and the quality of the fit is $\chi^2/\text{d.o.f.} = 17.4/6$.

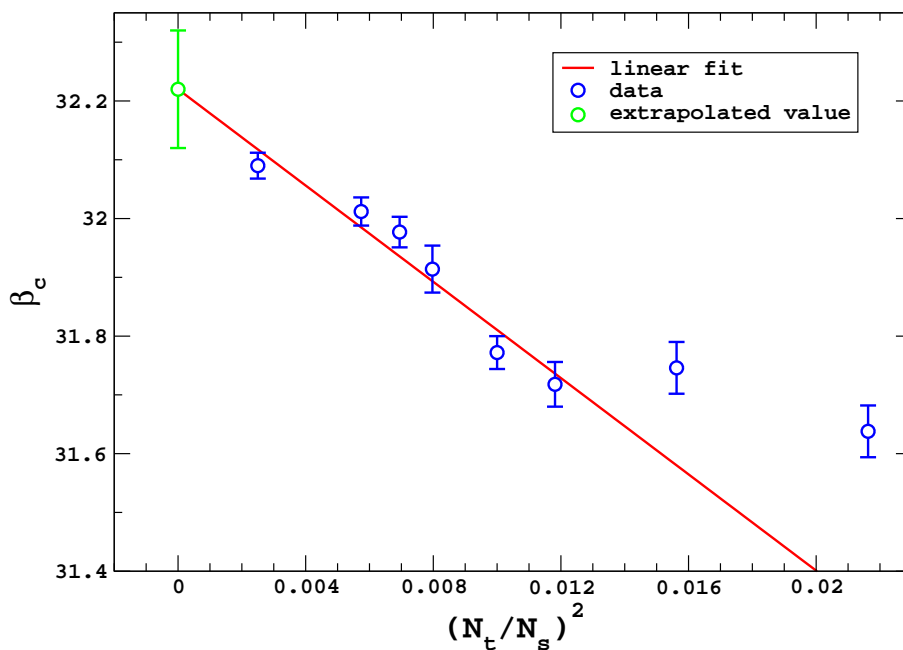


Figure 8: The same as in figures 6 and 7, this time for $N_t = 5$. For $N_t = 3$ and 4, the data show linear behavior for $(N_t/N_s)^2 \leq 0.015$. Hence, the two smallest N_s values are excluded from the optimal fit. The extrapolated value is $\beta_{c,N_t} = 32.22(8)$ and the quality of the fit is $\chi^2/\text{d.o.f.} = 7.8/4$.

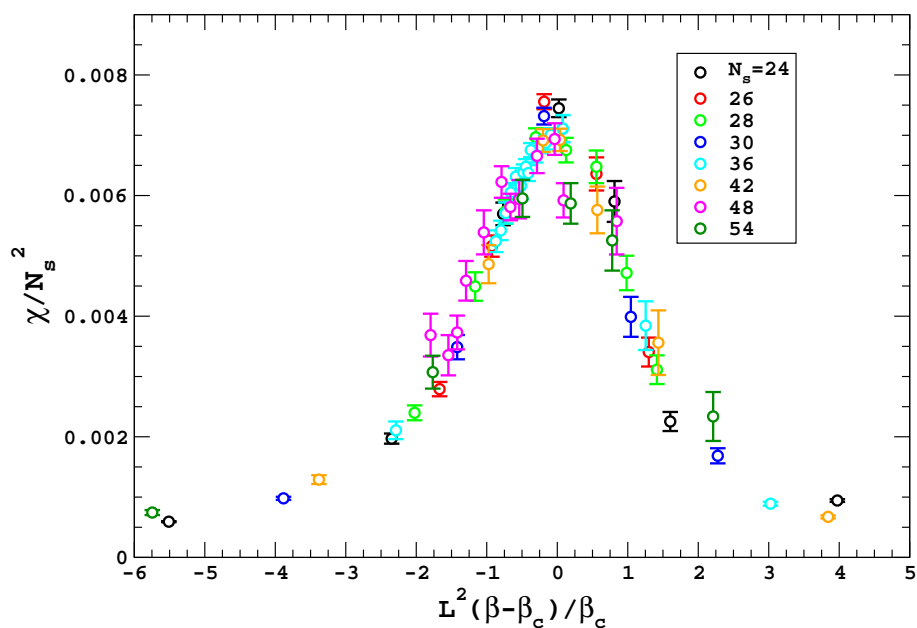


Figure 9: The rescaled Polyakov loop susceptibility χ/N_s^2 versus the finite-size scaling variable $L^2(\beta/\beta_c, N_s, N_t - 1)$ for $N_t = 3$, where $L = N_s/N_t$. The data fall onto a universal curve, consistent with first order exponents.

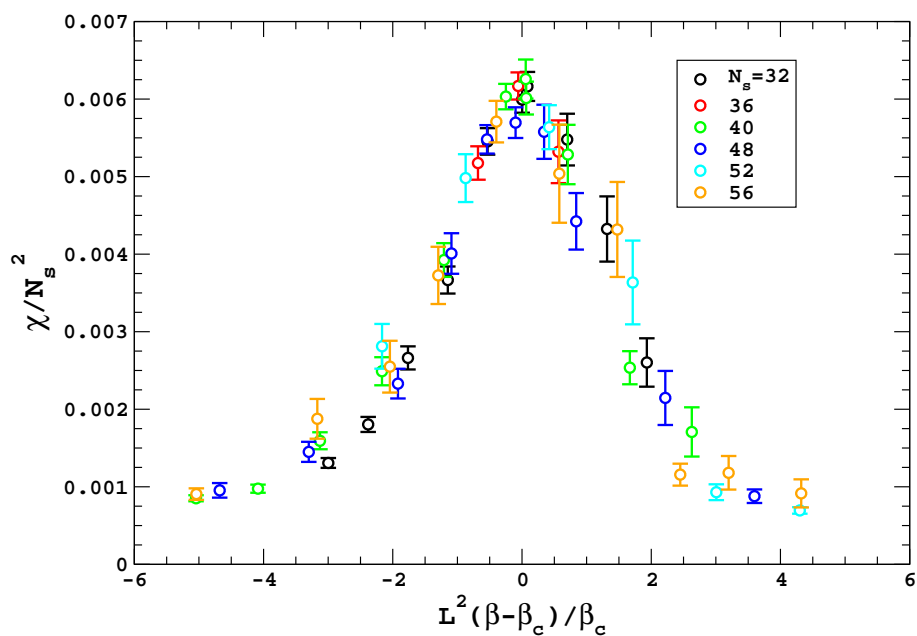


Figure 10: The same as in figure 9, this time for $N_t = 4$. Again, the data are consistent with first order exponents.

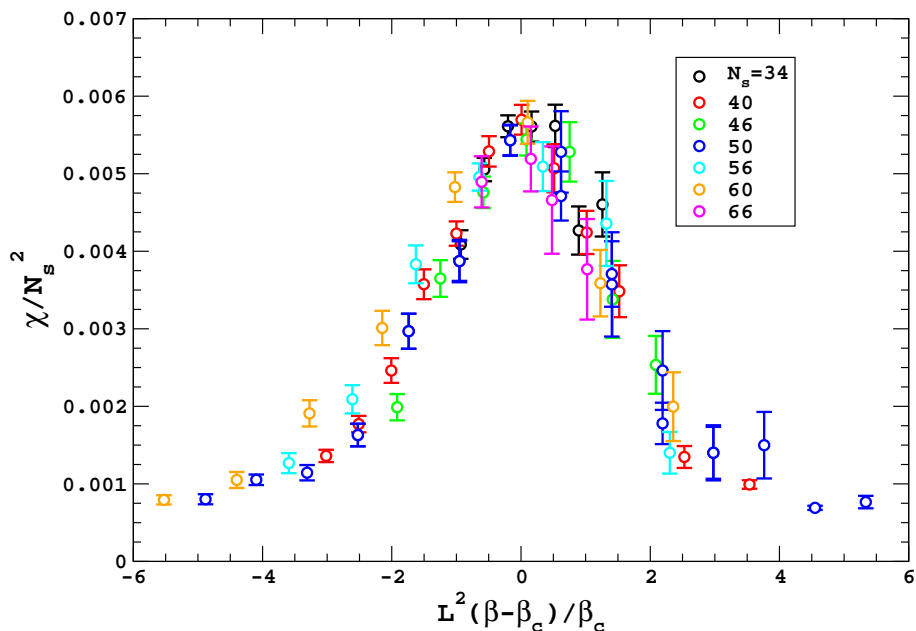


Figure 11: The same as in figures 9 and 10, this time for $N_t = 5$. Again, the data are consistent with first order exponents.

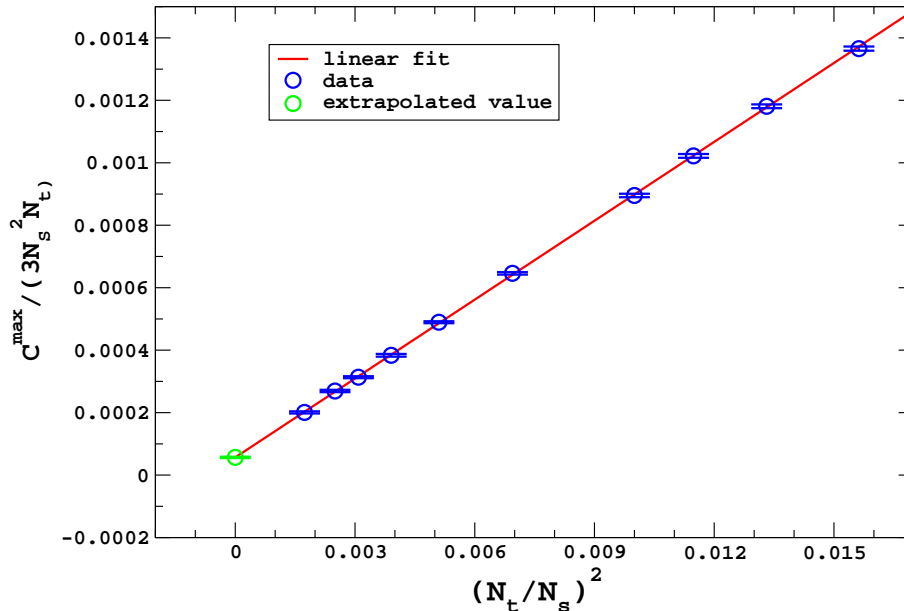


Figure 12: The peak of the rescaled specific heat capacity $C^{\max}/(3N_s^2 N_t)$ for $N_t = 3$ and a range of N_s . A linear fit describes the data very well and gives the extrapolated value $C^{\max}/(3N_s^2 N_t)(\infty) = 5.66(22) \times 10^{-5}$. The quality of the fit is $\chi^2/\text{d.o.f.} = 5.8/8$.

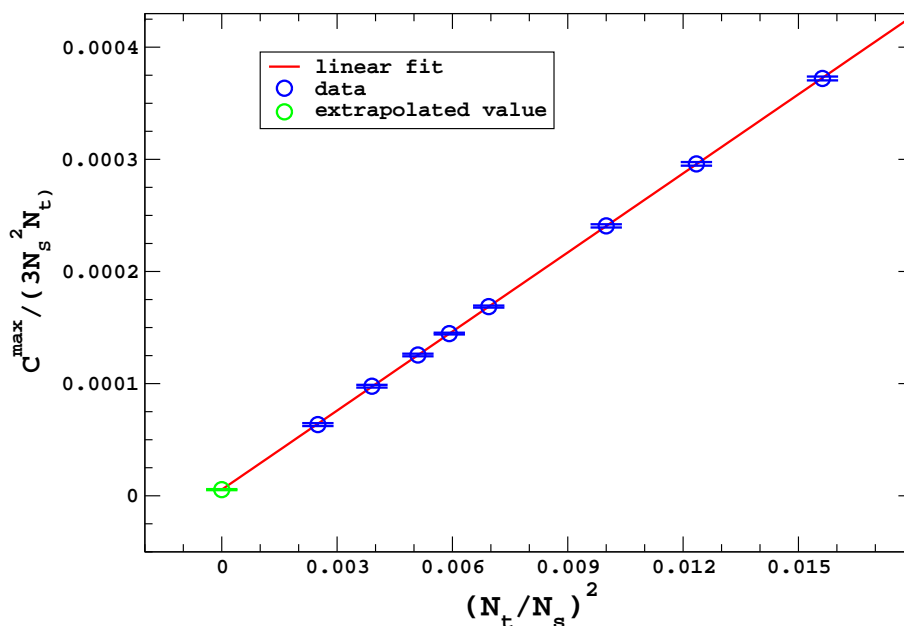


Figure 13: The same as in figure 12, this time for $N_t = 4$. Again, we linearly extrapolate to obtain $C^{\max}/(3N_s^2 N_t)(\infty) = 5.5(5) \times 10^{-6}$ and the quality of the fit is $\chi^2/\text{d.o.f.} = 0.65/6$.

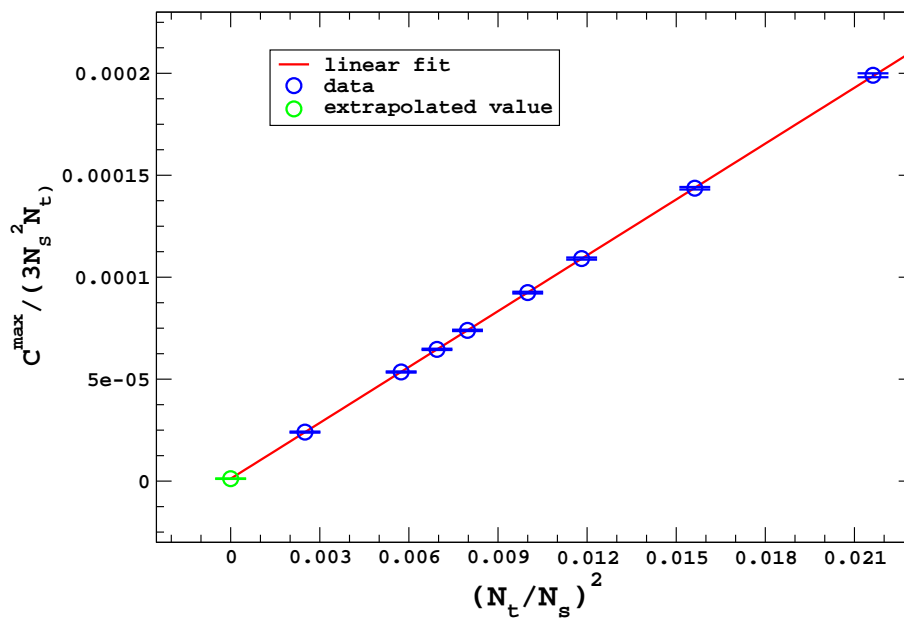


Figure 14: The same as in figures 12 and 13, this time for $N_t = 5$. A linear fit of all the data gives $C^{\max}/(3N_s^2 N_t)(\infty) = 1.19(11) \times 10^{-6}$ and the quality of the fit is $\chi^2/\text{d.o.f.} = 0.42/6$.

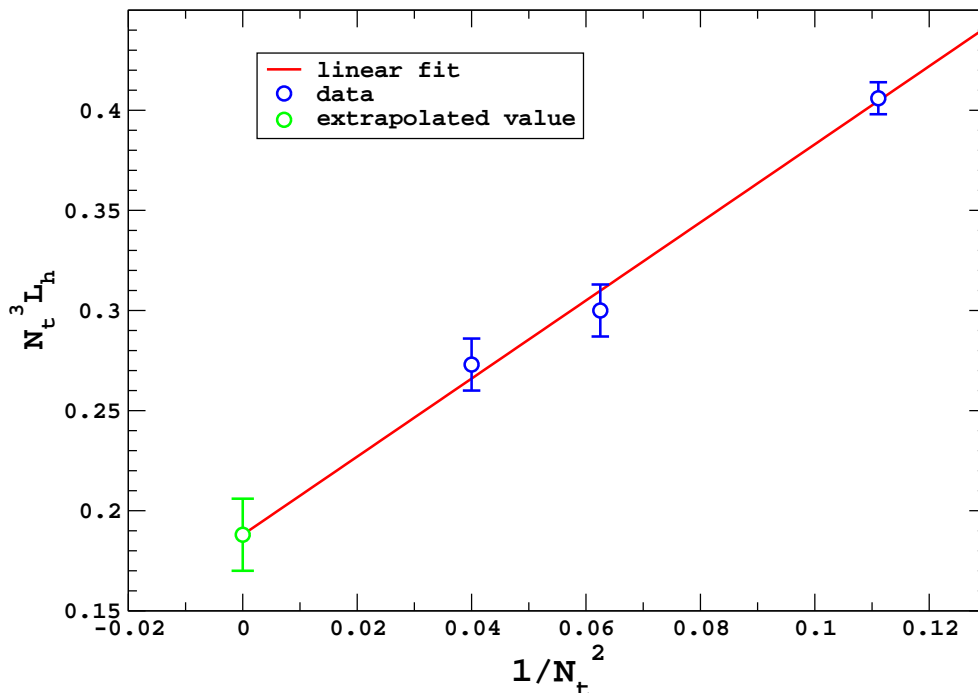


Figure 15: The latent heat L_h , taken from the $N_s \rightarrow \infty$ extrapolation of $C^{\max}/(3N_s^2 N_t)$. The data are extrapolated linearly in $1/N_t^2$. The continuum value is $N_t^3 L_h = 0.188(17)$ and the quality of the fit is $\chi^2/\text{d.o.f.} = 0.90/1$.

References

- [1] A.M. Polyakov, *Thermal properties of gauge fields and quark liberation*, *Phys. Lett.* **B 72** (1978) 477.
- [2] L. Susskind, *Lattice models of quark confinement at high temperature*, *Phys. Rev.* **D 20** (1979) 2610.
- [3] K. Holland and U.-J. Wiese, *The center symmetry and its spontaneous breakdown at high temperatures*, [hep-ph/0011193](#).
- [4] J. Kuti, *Lattice QCD and string theory*, PoS(LAT2005)001 [PoS(JHW2005)009].
- [5] J. Greensite, *The confinement problem in lattice gauge theory*, *Prog. Part. Nucl. Phys.* **51** (2003) 1.
- [6] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.
- [7] E. Witten, *Baryons in the $1/N$ expansion*, *Nucl. Phys.* **B 160** (1979) 57.
- [8] B. Svetitsky and L.G. Yaffe, *Critical behavior at finite temperature confinement transitions*, *Nucl. Phys.* **B 210** (1982) 423.
- [9] J. Kuti, J. Polonyi and K. Szlachanyi, *Monte Carlo study of SU(2) gauge theory at finite temperature*, *Phys. Lett.* **B 98** (1981) 199.
- [10] L.D. McLerran and B. Svetitsky, *Quark liberation at high temperature: a Monte Carlo study of SU(2) gauge theory*, *Phys. Rev.* **D 24** (1981) 450.

- [11] L.D. McLerran and B. Svetitsky, *A Monte Carlo study of SU(2) Yang-Mills theory at finite temperature*, *Phys. Lett.* **B 98** (1981) 195.
- [12] J. Engels, F. Karsch, H. Satz and I. Montvay, *High temperature SU(2) gluon matter on the lattice*, *Phys. Lett.* **B 101** (1981) 89.
- [13] R.V. Gvai, *The deconfinement transition in SU(2) lattice gauge theories*, *Nucl. Phys.* **B 215** (1983) 458.
- [14] R.V. Gvai, F. Karsch and H. Satz, *Universality in finite temperature lattice QCD*, *Nucl. Phys.* **B 220** (1983) 223.
- [15] J. Engels, J. Fingberg and M. Weber, *Finite size scaling analysis of SU(2) lattice gauge theory in (3 + 1)-dimensions*, *Nucl. Phys.* **B 332** (1990) 737.
- [16] J. Engels, J. Fingberg and D.E. Miller, *Phenomenological renormalization and scaling behavior of SU(2) lattice gauge theory*, *Nucl. Phys.* **B 387** (1992) 501.
- [17] T. Celik, J. Engels and H. Satz, *The order of the deconfinement transition in SU(3) Yang-Mills theory*, *Phys. Lett.* **B 125** (1983) 411.
- [18] J.B. Kogut et al., *Deconfinement and chiral symmetry restoration at finite temperatures in SU(2) and SU(3) gauge theories*, *Phys. Rev. Lett.* **50** (1983) 393.
- [19] S.A. Gottlieb et al., *The deconfining phase transition and the continuum limit of lattice quantum chromodynamics*, *Phys. Rev. Lett.* **55** (1985) 1958.
- [20] F.R. Brown, N.H. Christ, Y.F. Deng, M.S. Gao and T.J. Woch, *Nature of the deconfining phase transition in SU(3) lattice gauge theory*, *Phys. Rev. Lett.* **61** (1988) 2058.
- [21] M. Fukugita, M. Okawa and A. Ukawa, *Order of the deconfining phase transition in SU(3) lattice gauge theory*, *Phys. Rev. Lett.* **63** (1989) 1768.
- [22] N.A. Alves, B.A. Berg and S. Sanielevici, *Partition function zeros and the SU(3) deconfining phase transition*, *Phys. Rev. Lett.* **64** (1990) 3107.
- [23] M. Wingate and S. Ohta, *Deconfinement transition and string tensions in SU(4) Yang-Mills theory*, *Phys. Rev. D* **63** (2001) 094502 [[hep-lat/0006016](#)].
- [24] B. Lucini, M. Teper and U. Wenger, *The deconfinement transition in SU(N) gauge theories*, *Phys. Lett.* **B 545** (2002) 197 [[hep-lat/0206029](#)].
- [25] B. Lucini, M. Teper and U. Wenger, *The high temperature phase transition in SU(N) gauge theories*, *JHEP* **01** (2004) 061 [[hep-lat/0307017](#)].
- [26] B. Lucini, M. Teper and U. Wenger, *Properties of the deconfining phase transition in SU(N) gauge theories*, *JHEP* **02** (2005) 033 [[hep-lat/0502003](#)].
- [27] M. Teper, *The Finite temperature phase transition of SU(2) gauge fields in (2 + 1)-dimensions*, *Phys. Lett.* **B 313** (1993) 417.
- [28] J. Engels et al., *A study of finite temperature gauge theory in (2 + 1) dimensions*, *Nucl. Phys.* **53** (*Proc. Suppl.*) (1997) 420 [[hep-lat/9608099](#)].
- [29] J. Christensen, G. Thorleifsson, P.H. Damgaard and J.F. Wheeler, *Three-dimensional deconfinement transitions and conformal symmetry*, *Phys. Lett.* **B 276** (1992) 472.
- [30] J. Christensen, G. Thorleifsson, P.H. Damgaard and J.F. Wheeler, *Thermodynamics of SU(3) lattice gauge theory in (2 + 1)-dimensions*, *Nucl. Phys.* **B 374** (1992) 225.

- [31] K. Holland, M. Pepe and U.J. Wiese, *The deconfinement phase transition of $Sp(2)$ and $Sp(3)$ Yang-Mills theories in $2 + 1$ and $3 + 1$ dimensions*, *Nucl. Phys. B* **694** (2004) 35 [[hep-lat/0312022](#)].
- [32] K. Holland, *Another weak first order deconfinement transition: three-dimensional $SU(5)$ gauge theory*, *JHEP* **01** (2006) 023 [[hep-lat/0509041](#)].
- [33] J. Liddle and M. Teper, *The deconfining phase transition for $SU(N)$ theories in $2 + 1$ dimensions*, *PoS(LAT2005)*188.
- [34] M. Gross and J.F. Wheeler, *Deconfining transitions in $(2 + 1)$ -dimensional $SU(3)$ and $SU(4)$ gauge theories*, *Z. Physik C* **28** (1985) 471.
- [35] P. de Forcrand and O. Jahn, *Deconfinement transition in $2 + 1$ -dimensional $SU(4)$ lattice gauge theory*, *Nucl. Phys.* **129** (Proc. Suppl.) (2004) 709 [[hep-lat/0309153](#)].
- [36] K. Holland, P. Minkowski, M. Pepe and U.J. Wiese, *Exceptional confinement in G_2 gauge theory*, *Nucl. Phys. B* **668** (2003) 207 [[hep-lat/0302023](#)].
- [37] M. Pepe, *Confinement and the center of the gauge group*, *PoS(LAT2005)*017 [*Nucl. Phys.* **153** (Proc. Suppl.) (2006) 207] [[hep-lat/0510013](#)].
- [38] M. Pepe and U.J. Wiese, *Exceptional deconfinement in G_2 gauge theory*, *Nucl. Phys. B* **768** (2007) 21 [[hep-lat/0610076](#)].
- [39] G. Cossu, M. D'Elia, A. Di Giacomo, B. Lucini and C. Pica, *G_2 gauge theory at finite temperature*, *JHEP* **10** (2007) 100 [[arXiv:0709.0669](#)].
- [40] J. Greensite, K. Langfeld, Š. Olejník, H. Reinhardt and T. Tok, *Color screening, Casimir scaling and domain structure in G_2 and $SU(N)$ gauge theories*, *Phys. Rev. D* **75** (2007) 034501 [[hep-lat/0609050](#)].
- [41] A. Maas and Š. Olejník, *A first look at Landau-gauge propagators in G_2 Yang-Mills theory*, [arXiv:0711.1451](#).
- [42] N. Weiss, *The effective potential for the order parameter of gauge theories at finite temperature*, *Phys. Rev. D* **24** (1981) 475.
- [43] N. Weiss, *The Wilson line in finite temperature gauge theories*, *Phys. Rev. D* **25** (1982) 2667.
- [44] V.M. Belyaev and V.L. Eletsky, *Two loop free energy for finite temperature $SU(2)$ gauge theory in a constant external field*, *Z. Physik C* **45** (1990) 355 [*Sov. J. Nucl. Phys.* **51** (1990) 168].
- [45] R.D. Pisarski, *Effective theory of Wilson lines and deconfinement*, *Phys. Rev. D* **74** (2006) 121703 [[hep-ph/0608242](#)].
- [46] C. Hoyos-Badajoz, B. Lucini and A. Naqvi, *Confinement, screening and the center on $S^3 \times S^1$* , [arXiv:0711.0659](#).
- [47] A.M. Ferrenberg and R.H. Swendsen, *New Monte Carlo technique for studying phase transitions*, *Phys. Rev. Lett.* **61** (1988) 2635.
- [48] A.M. Ferrenberg and R.H. Swendsen, *Optimized Monte Carlo analysis*, *Phys. Rev. Lett.* **63** (1989) 1195.
- [49] M. Creutz, *Monte Carlo study of quantized $SU(2)$ gauge theory*, *Phys. Rev. D* **21** (1980) 2308.

- [50] S.L. Adler, *An overrelaxation method for the Monte Carlo evaluation of the partition function for multiquadratic actions*, *Phys. Rev.* **D 23** (1981) 2901.
- [51] S.L. Adler, *Overrelaxation algorithms for lattice field theories*, *Phys. Rev.* **D 37** (1988) 458.
- [52] M. Creutz, *Overrelaxation and Monte Carlo simulation*, *Phys. Rev.* **D 36** (1987) 515.
- [53] F.R. Brown and T.J. Woch, *Overrelaxed heat bath and Metropolis algorithms for accelerating pure gauge Monte Carlo calculations*, *Phys. Rev. Lett.* **58** (1987) 2394.
- [54] N. Cabibbo and E. Marinari, *A new method for updating $SU(N)$ matrices in computer simulations of gauge theories*, *Phys. Lett.* **B 119** (1982) 387.
- [55] J. Hove and A. Sudbo, *Criticality versus q in the $2 + 1$ -dimensional Z_q clock model*, *Phys. Rev.* **E 68** (2003) 046107 [[cond-mat/0212255](#)].